The math below presents the basic model of growth, complementing the spreadsheet we will look at in class. This version uses a Cobb-Douglas production function, which in fact works well enough that most economists don't bother with fancier functions. There are many extensions; “endogenous” growth models, for example, incorporate explicit feedback effects, such as from income to education to productivity. A series of recent models by Oded Galor and investigate demographic feedbacks; I like them. We keep things simple; all such elements remain exogenous.

Production:

In a closed economy output Y is equivalent to income Y = f (K, L). So let's choose an explicit function:

$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$. (K is "kapital" inputs, L is "labour" inputs, A is a technology parameter and α is the share of kapital income in total income. I drop t time hereafter.)

This function exhibits constant returns to scale (double both inputs K and L, and output Y doubles). It also has diminishing returns in each input: if you keep throwing more labour into a factory without increasing kapital, the productivity of labour falls. I include A_t as an explicit productivity factor.

First, let us rewrite in per labour terms by dividing by the labour variable. While nationalistic politicians may be concerned about the total size of an economy, our focus is how well-off people are.

$$
\begin{aligned}
Y/L &= A K^{\alpha} L^{1-\alpha}/L = A K^{\alpha} L^{1-\alpha} L^{-1} \\
&= A K^{\alpha} L^{1-\alpha-1} = A K^{\alpha} L^{-\alpha} = A K^{\alpha} / L^{\alpha} = A (K/L)^{\alpha}.
\end{aligned}
$$

Typically we then write lower-case italic variables to represent per labour variables (k = K/L and y=Y/L). In that case we write our function in an even simpler form:

$$y = A k^{\alpha}.$$

Income per capita is thus a function of kapital intensity, and suffers diminishing returns therein.

How about growth? Let savings S = sY, that is, a constant share of income is saved. That is a strong assumption, in that savings will vary with the age structure of the population (fairly obvious) and with the growth rate (not so obvious!), and with the business cycle (savings rates increase with higher than normal incomes). But over the course of a few years it varies little.

In our simple economy the only outlet for savings is (domestic) investment I, which does not mean the purchase of a piece of paper (such a stock certificate or the title to a piece of land) but rather the construction of new productive capacity, including human capital through worker training. In other words,

$$I = S = sY.$$

However, we need to subtract depreciation δ, as machinery and buildings wear out over time. Net investment is then I - δK (and skills are lost, including through retirement). [Check: does this really affect our model? Set it to zero in our spreadsheet and see!]

Hence:

$$\Delta K = I - \delta K = sY - \delta K.$$

Now this will generate an equilibrium of zero growth when sY = δK as kapital accumulation will cease as will growth. (The two are guaranteed to cross because of dimishing returns – Y and thus sY becomes flat as K rises, whereas depreciation keeps on increasing at the same linear rate so they must eventually cross.

See lecture notes and the excel spreadsheet – no graphics here.

Returns to inputs: What returns do labour and kapital receive as inputs? Here we need to look at the marginal products of labour and kapital, the marginal rate of change of output to inputs. Let's do K first:

MPK = dY/dK = d ($A_t K_t^{\alpha} L_t^{1-\alpha}$) / dK – that is, the first derivative.

But for an exponential function that's simple, as $d(x^n) = nx^{n-1}\,dx$. Hence we have:

rate of return on kapital = r = MPK = $\alpha\ A_t K_t^{\alpha-1} L_t^{1-\alpha} = \alpha\, Y/K$

and thus **total return to kapital** = rate times quantity = rK = αY.

similarly **rate of return on labour** = dY/dL = $(1-\alpha)\, A_t K_t^{\alpha} L_t^{-\alpha}$.

total return to labour = wL = (1-α)Y.

So in our simple economy kapital gets a share α of output and labour gets (1-α). This is only true if we have constant returns to scale, which is equivalent to having the exponents sum to 1.

Why bother? Ah, but think in terms of empirics! – how do we figure out the α for our equation? It's easy: we just go to the data to find out what share of returns go to which sector of the economy. For high-income economies, kapital and labour get it all (once we account for depreciation and certain taxes). In lower-income economies, agriculture is typically important, and rental income significant: landlords are not mere fixtures of "classic" literature (and of days living in student digs), but are a big share of the economy.

Empirics

Let's find another way to express the growth rate of Y, simply taking logs of our basic model:

$$\log Y = \log A + \alpha \log K + (1-\alpha) \log L$$

which we can rearrange in per capita terms:

$$\log Y - \log L = \log A + \alpha (\log K - \log L)$$

and now differentiate – (remember that d(log x) = dx/x):

$$dY/Y - dL/L = dA/A + \alpha (dK/K - dL/L).$$

But dY/Y = ΔY/Y is just another way of writing the rate of growth in percentage terms (since we can multiply both sides by 100). Rewriting with "g" indicating the growth rate then gives us:

$$g_Y = g_A + \alpha\, g_K + (1-\alpha)\, g_L = g_A + \alpha\, g_K - \alpha\, g_L + g_L \text{ so that}$$

$$g_Y - g_L = g_A + \alpha\, (g_K - g_L)\text{. Or, using our lower-case notation: } g_y = g_A + \alpha g_k.$$

This may look arcane, but it says that, absent productivity increases, kapital per labour has to grow for per capita incomes to rise. Furthermore, because of diminishing returns, we only get **α** units of output per labour. In other words, investing your way to prosperity gets harder and harder: as the kapital stock rises, growth will inevitably slow. Population growth adds to that difficulty, because some of investment goes to equip new workers rather than to boost the output (productivity) of existing workers. That's clear in the above equation, where an increase in the growth rate of labor to ***n*** would decrease growth per capita by α***n***.

If kapital accumulation is the source of China's growth, then it is only a matter of time – less than a decade – before they fall from double-digit growth to low single digits. Higher savings boosts the level of k while higher δ depreciation lowers it. But with our growth process, k always approaches a constant level, and modest changes in parameters don't result in a different time horizon, only in modestly different levels of equilibrium income.

Empirical findings:

So when we bring data to this framework, what do we find? In the US, Solow found that he could only account for about half of growth: the residual (our "A" term) accounted for a full half, given real-world observations of Y, K, L and α. Subsequent research since the 1950s – Dale Jorgenson at Harvard is the doyen of "growth accounting" and productivity studies – found that Solow's initial result was very robust: adding more variables (different types of kapital and labour, changing levels of education) and using fancier functional forms (with another parameter) did not change his basic story.

So what is "A"?

Economists who are honest with themselves admit that they don't know. But they have given it a name (particularly in studies of individual industries): **Total Factor Productivity** or **TFP**.

Part of represents errors in measuring variables and from using simple functions. But qualitatively not much, given empirical robustness. Instead the majority represents (partly by definition) **technical change**, which includes not only "hard" technology but also the adaptation of institutions to better turn inputs into output. In the old days – before the 20th century – our economy was dominated by small firms. That began to change around WWI with the rise of large business enterprises, General Motors after 1920, Sears & Roebuck even earlier, and with WWII a more centralized government structure that could coordinate the construction of a national highway system and regulate the growth of utilities while fostering R&D. And we have a generation of MBAs trained and socialized to work in this environment of budgets and planning, very different from the skills of market-oriented entrepreneurs.

Extensions and implications:

Two-sector models

Implicit in the Solow framework is that resources can be seamlessly reallocated across sectors of the economy. New investment can occur anywhere; labour can move without hindrance. Everyone pays the same price for inputs. Technology likewise affects everyone, and while there are great differences at the micro level of individual firms there is less variation at the (narrow) industry level and even less when we move to broad aggregates. That assumption is not plausible in developing countries such as China.

Hence it is useful to build multi-sector models in which factors are immobile across sectors. To do this, we have outputs from sectors being used by other sectors (goods flow fairly freely) while inputs are constrained by institutional limitations – costly human migration, weak banking systems that do a poor job of directing resources to small firms (farms!), and potentially different tax rates (farmers may face a tax on gross output, big companies on net income).

W. Arthur Lewis was the seminal figure – he was also the first Nobel Laureate from a developing economy, and the first minority. He posited a 2-sector economy in which only goods could flow between sectors. The non-agricultural (urban, industrial) sector sold goods to agriculture, perhaps (in Soviet-influenced economies) with a high "monopoly" markup or (equivalently) farm output was taxed more than industry. In addition, the income and price elasticity of food consumption is low, those for industrial goods are high. So incomes in "traditional" agriculture could truck along at subsistence levels, and any gains in productivity would accrue to the "modern" industrial sector. Over time the modern sector would grow, but initially it was too small to affect the whole economy. Eventually you might have a transition, when the "modern" sector grew enough to start soaking up labour from agriculture and allowing wages there to rise, until eventually the economy grew into developed, single-sector-model-suffices status. (Gustav Ranis and John Fei at Yale extended and formalized the model.)

If this sounds complicated – well, you get a lot of parameters. But if markets worked well, then why would the world on average be poor? Lewis and Ranis insisted on pushing beyond the "neoclassical" world of the Solow model (limitations Bob Solow himself readily admitted) to impose rigidities on the development process.

One key insight comes out this extension: the process of migration – of people away from farms, of kapital markets away from the mutlinational elite – is the key to development. As long as the agricultural sector remains in its own world, farmers will remain poor.

One final "why"? – why do farmers remain poor? The answer is a combination of productivity increases and the elasticity of demand. As new technology and new resources affect agriculture (fertilizer from the "modern" sector, transportation that expands markets), output rises. But demand does not, or at least not commensurate with the increases in output: after an initial move away from subsistence diets, people spend smaller and smaller shares of their incomes on food. With a constant farm population – and to be realistic, the rural population is generally rising – the marginal return to additional hours in farming falls. So unless there is a mass migration from farm to factory, incomes remain stagnant, certainly in relative if not absolute terms. Better access to kapital helps only a little, though it is central to the growth of local non-agricultural production and attendant jobs, and over time to purchasing farm equipment so that those who remain in farming enjoy the same benefits from high kapital-to-labour benefits of their urban compatriots.

The demographic dividend: A recent extension incorporates demographics, specifically the age structure of a population. After all, what we want to count are labor inputs, adjusted for experience, not population. We know that many countries went through a "baby boom" in the post-WWII era. Others merely saw a long-standing high level of fertility fall. In a growing population, each cohort is larger than its predecessor. If you stack them up in a graph with the youngest on the bottom, girls on the right and boys on the left, you get an inverted pyramid, though at young ages the base is a bit heavier (in a normal fertility regime, about there is at birth about a 5% excess of males) and leans right at the top (as women outlive men by about 6 years on average). In rapidly growing populations (parts of sub-Saharan Africa) over half the population is under age 15. So for every adult – I did not pick age 15 randomly – there is at least one dependent. But when the birth rate falls that dynamic shifts: with low fertility, below 2.1 children per woman, each cohort is smaller than its predecessor. Suddenly the number of dependents a worker needs to support starts to fall. Now if public health and nutrition are adequate, eventually the number of retirees will rise. But in the interim demographic transition can give a big boost to per capita income, because the share of active workers is very high. That's where China is at right now, and while the working age population will stabilize around 2015 and then begin to fall, that shift won't occur very rapidly and there is (or so our readings will argue) a pool of "surplus labor" in the countryside that means the effective labor force will be augmented by migration for years to come.

Endogenous growth: Recent research tries to build in feedback mechanisms, typically to human capital and to fertility but also to investment and TFP. They seek to answer what might "tip" an economy from a subsistence or low-growth scenario into one of self-sustaining growth. One attraction of such models is that they allow economists to show off their math skills. In addition, there are lots of ways to make things endogenous, so there is no end to models. Another was the development of new econometric tools that made it possible to move them from paper to computer. Last, there are new data sets that allow economists to make their mark, or at least get enough publications for tenure. Such data includes not just more details on education and demographics, but also "corruption" and ethnicity and shared languages and...

My own judgement is that such studies really don't tell us much that is new. Many of the findings are facile, "education matters," something for which development economists in the 1960s and 1970s had plenty of evidence. Furthermore, cross-national data are low in comparability and poor countries are unlikely to be measuring things very precisely. Hence it takes a real leap of faith to think that the empirical magnitude of such effects (the size of estimated coefficients) matter. Furthermore, good estimates take lots of data, but it is hard to get measurements for many variables more often than once a year. Hence you don't have many observations or degrees of freedom. So in practice econometric studies use only a few control variables. That is however convenient for graduate students, because it means that you can create almost unlimited variations of the basic model, and generate "new" findings. Most of those can be found in the informal graphical analysis of Hollis Chenery and various colleagues that dates to the late 1960s and 1970s. Such work couldn't get published today because they didn't use fancy regression tools. That doesn't mean it isn't insightful.